PARAMETRIC INSTABILITY OF THE m = 1 MODE IN THE DYNAMIC STABILIZATION OF A PLASMA FILAMENT

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We consider the parametric excitation of long-wave magnetohydrodynamic oscillations of the m = 1 type in a cylindrical plasma conductor with an alternating, longitudinal, high-frequency current. The plasma cylinder is placed in a constant longitudinal magnetic field and is enclosed in a conducting case. The problem is solved on the basis of the flexible filament model under the assumption of ideal conductivity of the plasma and the case. Hill's method is used to study the stability of the equation with periodic coefficients that describe the oscillations of the filament. Results of computer calculations of the stability increments of oscillations in the first four resonance zones for various values of the parameters of the system are given.

As was shown in [1], when a high-frequency alternating current is used for the dynamic stabilization of a plasma cylinder in a longitudinal magnetic field, parametric excitation of magnetohydrodynamic proper oscillations of the cylinder, that are characterized by the azimuthal wave number m = 1, can occur. In [2] the boundaries of the first two zones of parametric excitation of the m = 1 mode were determined, and an analytic expression for the maximum increment of the buildup of short-wave oscillations was obtained ($ka \gg 1$, where k is the wave number of the perturbation and a is the radius of the cylinder). It is of interest to make a more detailed investigation of the instability in question in the range of long-wave perturbations ($ka \lesssim 1$), which must be excited under experimental conditions (see, for example, [3, 4]). This problem is solved in the present paper. Here we give a numerical calculation of the instability increments of long-wave oscillations of the m = 1 type, excited in a cylindrical plasma conductor by a high-frequency longitudinal current. In contrast to [1, 2], the effect of the conducting case, surrounding the plasma cylinder, is taken into account. The problem is solved on the basis of the flexible filament model under the assumption of ideal conductivity of the plasma and the case. Hill's method [5] is used to investigate the stability of the equation with periodic coefficients that describes the oscillations of the filament. Various possible regimes of the operation of the system are considered.

1. Formulation of the Problem

Suppose that a high-frequency longitudinal current $I = I_0 \cos \omega t$ flows along the surface of a cylindrical plasma conductor. The conductor is placed in a constant longitudinal magnetic field, equal to B_e outside and to B_i inside the plasma, and is enclosed in a conducting case of radius b. The plasma pressure p counterbalances the time-averaged pressure of the magnetic field

$$8\pi p = B_e^2 - B_i^2 + \langle B_a^2 \rangle \tag{1.1}$$

Here $B_a = B_{a0} \cos \omega t = 2I/ca$ is the azimuthal field of the current I on the surface of the filament and the angular brackets denote a time average.

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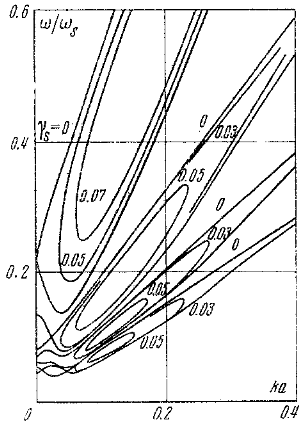

Fig. 1

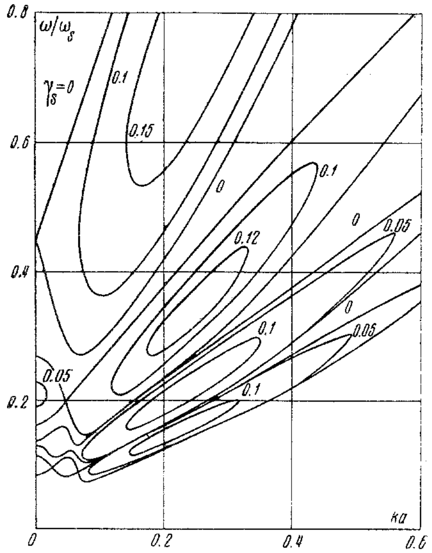

Fig. 2

It is convenient to reduce the investigation of the stability of the system under consideration with respect to magnetohydrodynamic perturbations of the m = 1 type, for which the surface of the conductor is described by the equation

$$r = a + \xi_1(t)\exp(ikz \pm i\theta) \quad (\xi_1 \ll a), \tag{1.2}$$

to the flexible filament model. The force F per unit length, perpendicular to the z axis, acting on the perturbed conductor, is calculated in the magnetostatic approximation, and then we consider the equation for the transverse motion of a length element of the conductor with mass M per unit length

$$M d^2\xi_1 / dt^2 = F \tag{1.3}$$

in which the time dependence of the magnetic field, entering into F, is already explicitly taken into account. A comparison with results of a rigorous magnetohydrodynamic analysis (valid for the case of static fields) shows that this model describes the system under consideration sufficiently well if the plasma is considered to be incompressible and the perturbations are long-wave ones.

In the case under consideration the force F can be found from formulas of [6] and has the form

$$F = -\tfrac{1}{4}\left[\alpha_e^\circ (kaB_e \pm B_a)^2 + \alpha_i (ka)^2 B_i^2 - B_a^2\right]\xi_1 \tag{1.4}$$

Here

$$\alpha_e^\circ = \frac{\alpha_e + \alpha_i\delta}{1-\delta}, \qquad \alpha_e = -\frac{K_1(ka)}{kaK_1'(ka)}$$

$$\alpha_i = \frac{I_1(ka)}{kaI_1'(ka)}, \qquad \delta = \frac{K_1'(ka)\,I_1'(kb)}{I_1'(ka)\,K_1'(kb)}$$

$K_1(x)$, $I_1(x)$ are modified Bessel functions, and a dash denotes differentiation with respect to the argument. When the case is absent, $\delta = 0$ and (1.4) reduces to the corresponding expression in [2].

Because of the sinusoidal time dependence of the current I assumed above, Eq. (1.3) is an equation with periodic coefficients. Upon the substitution $\omega t = 2\tau$ it reduces to the standard form of Hill's equation [5] with three terms:

$$d^2\xi_1 / d\tau^2 + (\theta_0 + 2\theta_1 \cos 2\tau + 2\theta_2 \cos 4\tau)\,\xi_1 = 0 \tag{1.5}$$

Here

$$\theta_0 = 4(\omega_s/\omega)^2\left[(ka)^2(\alpha_e^\circ + \alpha_i h_i^2) + \tfrac{1}{2}(\alpha_e^\circ - 1)h_a^2\right]$$

$$\theta_1 = \pm 4(\omega_s/\omega)^2 ka\alpha_e^\circ h_a, \quad \theta_2 = 2(\omega_s/\omega)^2(\alpha_e^\circ - 1)h_a^2$$

$$\omega_s = v_s/a, \quad v_s = B_e/\sqrt{4\pi\rho} \ \text{(where } \rho \text{ is the plasma density)},$$

$$h_i = B_i/B_e, \quad h_a = B_{a0}/B_e$$

We note that, in view of (1.1), the velocity v_s under typical experimental conditions when $\langle B_a^2\rangle \ll B_e$ is close to the velocity of magnetic sound in the plasma.

In the region $ka \lesssim (a/b)h_a$, where the influence of the case is considerable, the coefficients of Eq. (1.5) differ significantly from the coefficients of the analogous equation of [2]. In particular, $\theta_2 \gtrsim \theta_1$, so that the term with θ_2 in (1.5) cannot be neglected in our investigation of stability, as was done in [2].

We write the general solution of Eq. (1.5) in the form [5]

$$\xi_1(\tau) = C_1 e^{\mu\tau}\varphi_1(\tau) + C_2 e^{-\mu\tau}\varphi_2(\tau)$$

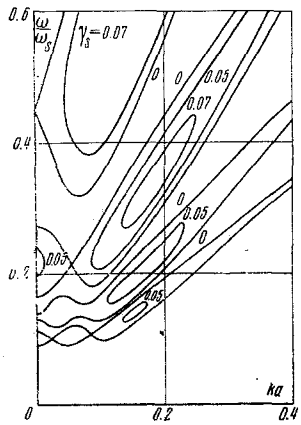

Fig. 3

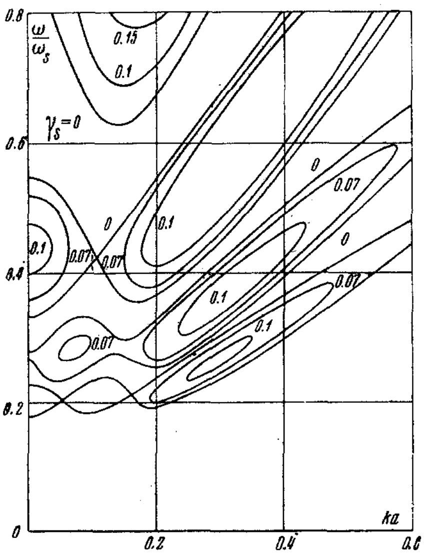

Fig. 4

where $\varphi_1(\tau)$, $\varphi_2(\tau)$ are periodic functions. Following Hill's method [5], we can find the complex characteristic exponent μ from the equation

$$\sin^2(^1/_2\pi i\mu) = \Delta \sin^2(^1/_2\pi\sqrt{\theta_0}) \tag{1.6}$$

which contains the infinite determinant $\Delta \equiv [A_{m,n}]$, whose elements, reckoned from the center, are equal to

$$A_{m,m} = 1, \quad A_{m,n} = \frac{\theta_{|m-n|}}{\theta_0 - 4m^2} \qquad (m \neq n)$$

In Eq. (1.5) $\theta_i \equiv 0$ for $i > 2$. The instability increment γ is evidently equal to $(\omega/2)\,\mathrm{Re}\,\mu$. Using (1.6), it is not difficult to show that for the n-th resonance zone γ is determined from the equation

$$\mathrm{ch}(2\pi\gamma/\omega) = (-1)^n(1 - 2\Delta\sin^2(^1/_2\pi\sqrt{\theta_0})) \tag{1.7}$$

Solution of Eq. (1.7) for values of n ranging from 1 to 4 and for various values of the parameters h_a, h_i, b/a was carried out by computer. The rank of the determinant $[A_{m,n}]$ was chosen to be equal to 11, which ensured a sufficient degree of accuracy in the calculations. In the plane of the variables ka, ω/ω_s the lines of equal values of the dimensionless increment $\gamma_s = \gamma/\omega_s$ were determined for each group of parameters. The variables ka, ω/ω_s were varied between the limits $0 \leq (ka, \omega/\omega_s) \lesssim 1$. This range is of greatest interest from a practical point of view since, in the first place, long-wave perturbations have the greatest instability increment [2], and, in the second place, the frequency ω_s under typical experimental conditions is of the order $(0.5\text{–}1)\cdot 10^8\ \mathrm{sec}^{-1}$, while ω usually does not exceed $10^7\ \mathrm{sec}^{-1}$.

2. Discussion of the Results

Results of the calculations are shown in Figs. 1-5. Equal-increment lines for a high-frequency z-pinch ($h_i = 1$) are plotted in Figs. 1-4, with the following parameters: $h_a = 0.25$, $b/a = 2.5$ (Fig. 1); $h_a = 0.5$, $b/a = 2.5$ (Fig. 2); $h_a = 0.25$, $b/a = 1.5$ (Fig. 3); $h_a = 0.5$, $b/a = 1.5$ (Fig. 4). Equal-increment lines for a system of the θ-pinch type ($h_i = 0$) are plotted in Fig. 5, with the parameters $h_a = 0.25$, $b/a = 5$. In each resonance zone the largest of the values of the increment γ_s indicated on the graphs are close to the maximum increment in the zone in question. For an actual filament having a finite length L, the excitation zones split up into a series of vertical segments, corresponding to discrete values of the dimensionless wave number

$$ka = (2\pi a/L)\,j \quad (j = 0, 1, 2, \ldots)$$

We shall study point out some characteristic properties of the instability under study, which are reflected in Figs. 1-5. First of all, we note that the calculated increments γ_s do not exceed the maximum instability increment of the m = 1 mode in the case $I = I_0$ = const (Shafranov–Kruskal mode), which is attained for $ka \approx h_a$ and in dimensionless form is equal to $\gamma_{SK} \approx \gamma_s$. The conducting case has a substantial influence on the parametric excitation of the m = 1 mode, increasing the rigidity of the current-conducting filament to long-wave bends and displacements. The most important result of this effect is that in the region $ka < (a/b)h_a$ the resonance zones are displaced toward higher frequencies, approximately proportional to $(b/a)h_a$. In particular, the possibility of buildup of perturbations with k = 0 appears (displacement of the filament as a whole), as compared with a filament without the case, which is neutrally stable. When the case is present there is also a change in the shape of the resonance zones, in which deformations in the form of constrictions appear. The latter effect is due to the interaction of modulation harmonics with the frequencies ω and 2ω, which, as calculations show, are equal in order of magnitude to the amplitudes θ_1 and θ_2 in the region $ka \lesssim (a/b)^2 h_a$.

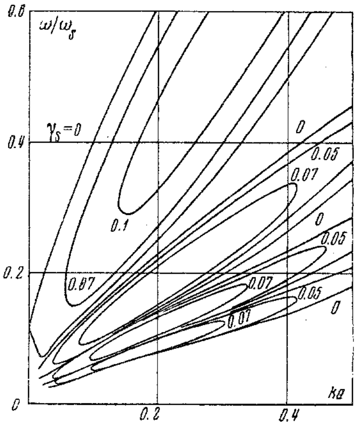

Fig. 5

The first resonance zone represents the greatest danger to the stability of the filament. The fundamental oscillation in this zone has the frequency $\omega/2$ (in the n-th resonance zone the oscillation with the frequency $n\omega/2$ has the maximum amplitude). The limiting value $(\omega/\omega_s)_{min}$, below which the first zone does not descend, is given approximately by the expression

$$\left(\frac{\omega}{\omega_s}\right)_{\min} \approx \frac{\sqrt{2}\, h_a}{\sqrt{(b/a)^2 - 1}}$$

Thus, with a sufficiently tight case at frequencies $\omega \ll \omega_s$, the buildup of the most rapidly growing oscillations with the frequencies $n\omega/2$, that are excited in the other zones, have increments that are considerably smaller. Elevation of the resonance zones also occurs with an increase in the parameter h_a; however, in this case the resonance zones spread out, and the instability increments increase.

We also remark that for systems of the z-pinch type ($h_i = 1$, Figs. 1-4) parametric excitation of oscillations with a given wave number k occurs for higher relative frequencies ω/ω_s than for systems of the θ-pinch type ($h_i = 0$, Fig. 5). This effect is due to the increase in the rigidity of the filament because of entrainment of the plasma by the field B_i.

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